Path-Constrained Traffic Assignment

Model and Algorithm

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This paper presents a mathematical programming model and solution method for the path-constrained traffic assignment problem, in which route choices simultaneously follow the Wardropian equilibrium principle and yield the distance constraint imposed on the path. This problem is motivated by the need for modeling distance-restrained electric vehicles in congested networks, but the resulting model and solution method can be applied to various conditions with similar path-based constraints. The equilibrium conditions of the problem reveal that any path cost in the network is the sum of corresponding link costs and a path-specific out-of-range penalty term. The suggested method, based on the classic Frank–Wolfe algorithm, incorporates an efficient constrained shortest-path algorithm as its subroutine. This algorithm fully exploits the underlying network structure of the problem and is relatively easy to implement. Numerical results from the examples of problems provided show how the equilibrium conditions are reshaped by the path constraint and how the traffic flow patterns are affected by different constraint tightness levels.

Since the seminal work of Beckmann et al. (*1*), various traffic assignment problems (TAPs) have been proposed and formulated for characterization and prediction of traffic flow patterns on road networks on which congestion effects become a prominent phenomenon as a result of the flow–cost interaction. Traffic flow patterns are often determined according to the tendency of a traffic network to move toward a stable state or a prescribed performance criterion. The two most commonly used network assignment principles, namely, user equilibrium and system optimum, are generally attributed to Wardrop (*2*).

The Wardropian user equilibrium principle is based on the individual self-optimal assumption; that is, each traveler seeks a route to minimize his or her own travel cost, and a stable condition is reached when no traveler can reduce his or her travel cost by unilaterally changing routes. This paper describes a study of a special TAP of the user equilibrium type with extra constraints: the path-constrained TAP. In its simplest form, two types of constraints may be imposed on paths, namely, path flow constraint and path cost constraint. A path cost could be flow dependent, such as travel time and reliability,

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or flow independent, such as distance. For urban commuters who need to arrive at their working place in a certain amount of time, their route choices are subject to a flow-dependent time constraint, whereas for electric vehicle drivers, their route choices are limited to those paths along which the length is less than the driving range limit confined by the battery capacity.

The focus of this paper is on the distance-constrained TAP (DCTAP), in which distance is used as the proxy of a class of flow-independent path attributes. The distance-constrained traffic equilibrium is defined as an extension of the Wardropian user equilibrium principle: each traveler seeks a route to minimize his or her own travel cost subject to maximum path length; no traveler can reduce his or her travel cost by unilaterally changing routes from the distance-restrained route set.

PROBLEM MOTIVATION AND STATEMENT

Plug-in electric vehicles (PEVs) rely primarily or exclusively on electricity and are designed to be recharged by the electricity grid. Those PEVs that rely entirely on electricity (e.g., the 2011 Nissan Leaf and the 2012 Mitsubishi i) are defined as battery electric vehicles (BEVs). Because the electricity consumption is typically proportional to the driving distance, the battery capacity confines a driving range for BEVs. For example, the driving range of a Mitsubishi i vehicle is about 60 mi, and a Nissan Leaf vehicle can run up to 100 mi on a single charge. Once a BEV's battery storage is depleted, it cannot be driven any farther. This distance limit inevitably changes BEV drivers' travel behaviors. Given the scarce availability of charging stations in the current market, consumers perceive this driving range limit to be a potential worry, or it may cause, as it is called by some researchers, "range anxiety": the fear of being stranded because the battery runs out of charge (*3*).

The massive adoption of PEVs requires fundamental changes to the existing travel demand and network flow models to capture changing or induced behaviors and constraints properly, especially in urban areas where BEVs may dominate the passenger car population in the future. On the basis of the current battery technology, charging of a BEV battery can take several to more than 10 hours. More advanced battery-charging technology is still under development, and the distribution of the existing charging infrastructure is far below the level required to provide a minimum coverage. For example, among all the states in the United States, California is the only state that now has more than 50 charging stations (*4*). Although many cities are planning the construction and expansion of public charging infrastructures, it is currently likely that most BEV drivers will need to charge their vehicles at home (*5*).

This paper considers the proposed DCTAP as a pure mathematical tool and a fundamental modeling device that can be used to

characterize and mimic aggregate BEVs' travel behavior in networks with a given set of ideal socioeconomic assumptions and restrictions. The authors, however, do not claim the applicability and suitability of the defined problem and model for accurate quantification of network flows and congestion levels. One must consider individual travelers or traffic flows with alternative travel behavior flexibilities and restrictions because of the various vehicle technologies and fueling infrastructures. A more general situation is that vehicles of other technological types, such as conventional gasoline vehicles and emerging hybrid vehicles, prevail in the network; a practical operational model must accommodate mixed flows of these different types of vehicles. Despite these realistic situations, however, the authors believe that the simple model and solution method presented in this paper provide a theoretical basis and behavioral insights for construction of more complex models with realistic considerations.

RELEVANT MODELS

To the best of the authors' knowledge, no research about the DCTAP has been done before. The DCTAP can be simply described as the classic TAP with a side constraint for each path in the network. Research efforts on extension of the basic TAP by addition of side constraints can be found in the literature. A typical example is the capacity-constrained TAP, which imposes upper bounds on link flows (*6–8*). Larsson and Patriksson extended the capacity-constrained TAP to a general side-constrained TAP, but they assumed that the general side constraints involve only the link flows (*9*).

Another related research stream is the bicriteria TAP. Dial (*10–12*) and Leurent (*13*) developed bicriteria TAP models in which both routing criteria are flow dependent. A bicriteria or multicriteria TAP model enables one to represent disaggregate trade-offs between different criteria in the trip-makers' route choices, for example, travel time and monetary cost. However, the distance limit considered in this work is a hard constraint, which requires that it be treated as a constraint rather than an objective criterion.

RELEVANT ALGORITHMS

It is well recognized that the standard TAP can be solved efficiently with a solution algorithm of linear approximation like the Frank–Wolfe method (*14*). It is popular because of its relative simple algorithmic structure, ease of implementation, and availability in commercial software packages. This method is especially efficient for determination of the network equilibrium flows because the linearized subproblem calls for minimization of the total travel cost over a network with fixed link travel costs, which actually collapses to a shortest-path problem for each origin–destination (O-D) pair. The shortest-path problem can be solved efficiently by polynomial algorithms such as the algorithm of Dijkstra (*15*) or Floyd–Warshall (*16, 17*).

In the case of a TAP with link-level side constraints, the efficiency of a Frank–Wolfe-type algorithm may be seriously degraded when the linearized subproblem is not separable for each O-D pair. In the special case of capacity-constrained TAP, for example, the subproblem becomes a multicommodity minimum-cost flow problem, which is quite expensive to solve repeatedly. When the side constraints involve only link flows, however, the side constraints can be relaxed through dual or penalty schemes and the relaxed problem is merely a basic TAP with generalized link travel costs. Any efficient algorithm for solving the basic TAP can be used to solve this relaxed problem. For this reason, the dual or penalty scheme is a popular approach for solving this type of problem.

However, when the side constraints involve path flows, like what the authors studied in this work, the equilibrium conditions of the relaxed problem are based on the path impedance, which is the sum of path travel cost and a path-specific cost term (see the discussion below). In other words, if the DCTAP is relaxed into a sequence of unconstrained problems through a dual or penalty scheme, the relaxed subproblems are TAPs with path-specific costs that are much more difficult to solve than the basic TAP. Gabriel and Bernstein studied the TAP with nonadditive path costs and proposed a general functional form of nonadditive path costs that includes the case of path-specific costs (*18*). A generic algorithm based on sequential quadratic programming was proposed to solve it, but no numerical test result was given. Lo and Chen used a smooth gap function to convert the nonlinear complementarity problem formulation for the TAP with nonadditive costs to an equivalent unconstrained optimization problem and proposed two solution approaches to solve it (*19*). Their first approach involves enumeration of a predefined set of routes in advance, and the second one resorts to solution of the *k*th shortest-path problem repeatedly. Neither of these two methods, however, is applicable for large networks. In the case described here, the path-specific cost term involves the dual variable or Lagrangian multiplier associated with the side constraint, which is not explicitly known. The values of Lagrangian multipliers must be iteratively derived, and a TAP with path-specific costs must be solved. This solution strategy does not seem to be efficient, given that such a TAP with path-specific costs itself is not readily tackled.

Another difference between the link-level constrained problem and the path-level constrained problem is that in the linearization scheme, as in the Frank–Wolfe algorithm, the linearized subproblem of the path side-constrained problem can still be solved separately for each O-D pair. The authors will actually prove here that the linearized subproblem for DCTAP is a constrained shortest-path problem (CSPP), that, however, is NP-complete in its worst case (*20*) and can be solved relatively efficiently in practice. Therefore, instead of the dual strategy, the linearization strategy, specifically, the Frank–Wolfe framework, is applied to solve the DCTAP.

CSPP can be stated as follows: given a directed network that has a cost and a resource associated with each link, find the least-cost path between two specified nodes such that the total resource consumed by this path is less than a prespecified limit. The exact algorithms for the solution of CSPP that have been developed can be divided into two main categories: one involves solution of a relaxed problem by Lagrangian or linear relaxation (*21–24*), and the other uses labeling methods based on dynamic programming (*25–27*). Among these methods, the label-setting algorithm developed by Desrochers and Soumis (*26*) appears to have been widely regarded as an efficient one. A recent improvement on the algorithm implementation was made by Dumitrescu and Boland (*27*), who modified the label-setting algorithm of Desrochers and Soumis (*26*) and combined it with a preprocessing procedure. They found that although it had the same worst-case complexity, the modified label-setting algorithm together with the preprocessing procedure performs markedly better than the original label-setting algorithm in numerical tests, with test problems with thousands of nodes being solved in seconds. For details about the labeling algorithm and preprocessing procedure, interested readers are referred to the report and references therein of Dumitrescu and Boland (*27*).

OUTLINE

The remainder of this paper is organized as follows. The DCTAP model is first elaborated; in particular, the model formulation and solution properties are discussed. Next, the authors show that this problem can be solved by the Frank–Wolfe algorithm with an implementation of the CSPP as a subroutine. Computational results for a couple of test problems of small and medium sizes are given. The results from these numerical tests justify the applicability of the model and solution method. The means by which network flow patterns are affected by different distance limits are also demonstrated by use of the model and method as a tool. Finally, directions for future exploration are discussed.

MODEL FORMULATION AND PROPERTIES

Model Formulation

As noted above, the classic TAP can be described by an equivalent mathematical program that is known as Beckmann's transformation. By use of this basic formulation and incorporation of different distance constraints for different makes and classes of vehicles, the following DCTAP is proposed:

$$\min Z(\mathbf{x}(\mathbf{f})) = \sum_a \int_0^{x_a} t_a(\omega)\,d\omega \tag{1}$$

subject to

$$\sum_k f_{k,m}^{rs} = q_m^{rs} \qquad \forall r, s, m \tag{2}$$

$$f_{k,m}^{rs} \geq 0 \qquad \forall k, r, s, m \tag{3}$$

$$\left(D_m - l_k^{rs}\right) f_{k,m}^{rs} \geq 0 \qquad \forall k, r, s, m \tag{4}$$

$$x_a = \sum_{rs}\sum_k\sum_m f_{k,m}^{rs}\delta_{a,k}^{rs} \qquad \forall a \tag{5}$$

where

> x_a = traffic flow rate on link a,
> $t_a(\cdot)$ = travel cost of link a,
> $f_{k,m}^{rs}$ = traffic flow rate of mth class of vehicles on path k from origin r to destination s,
> q_m^{rs} = travel demand rate of mth class of vehicles from origin r to destination s,
> D_m = distance limit of mth class of vehicles, and
> l_k^{rs} = the length of path k from origin r to destination s, $\sum_a d_a \delta_{a,k}^{rs}$, where d_a is distance of link a and $\delta_{a,k}^{rs}$ is link–path incidence parameter; $\delta_{a,k}^{rs}$ is equal to 1 if link a is contained in kth path from origin r to destination s; otherwise, $\delta_{a,k}^{rs}$ is equal to 0.

In this model, Objective Function 1 and Constraints 2 and 3 constitute the basic TAP model. Inequality 4 is the added side constraint that has a direct physical interpretation: if the flow rate of the mth class of vehicles on path k connecting origin r and destination s is positive, the length of this path cannot exceed D_m, the distance limit of the mth class of vehicles. For gasoline vehicles, the distance limit is infinity. Therefore, this model takes into account both BEVs and gasoline vehicles.

Optimality Conditions

To check the equilibrium conditions corresponding to the point at which the mathematical program described above is minimized, the first-order conditions of the program are analyzed. Let u_m^{rs} and $\lambda_{k,m}^{rs}$ denote the dual variables associated with Equations 2 and 4, respectively. If u_m^{rs} is unrestricted in sign, and $\lambda_{k,m}^{rs}$ is restricted to be nonnegative, the Lagrangian problem can be written as

$$L(\mathbf{f}, \mathbf{u}, \lambda) = Z(\mathbf{x}(\mathbf{f})) + \sum_{rs}\sum_m u_m^{rs}\left(q_m^{rs} - \sum_k f_{k,m}^{rs}\right)$$
$$- \sum_{rs}\sum_k\sum_m \lambda_{k,m}^{rs}\left(D_m - l_k^{rs}\right) f_{k,m}^{rs} \tag{6}$$

$$f_{k,m}^{rs} \geq 0 \qquad \forall k, r, s, m \tag{7}$$

where dual variables $\lambda_{k,m}^{rs} \geq 0$.

The optimality conditions of the Lagrangian problem are

$$f_{k,m}^{rs*}\left[\sum_a t_a \delta_{a,k}^{rs} - \lambda_{k,m}^{rs*}\left(D_m - l_k^{rs}\right) - u_m^{rs*}\right] = 0 \qquad \forall k, r, s, m \tag{8}$$

$$\sum_a t_a \delta_{a,k}^{rs} - \lambda_{k,m}^{rs*}\left(D_m - l_k^{rs}\right) - u_m^{rs*} \geq 0 \qquad \forall k, r, s, m \tag{9}$$

$$f_{k,m}^{rs*} \geq 0 \qquad \forall k, r, s, m \tag{10}$$

$$\lambda_{k,m}^{rs*}\left(D_m - l_k^{rs}\right) f_{k,m}^{rs*} = 0 \qquad \forall k, r, s, m \tag{11}$$

$$\left(D_m - l_k^{rs}\right) f_{k,m}^{rs*} \geq 0 \qquad \forall k, r, s, m \tag{12}$$

$$\lambda_{k,m}^{rs*} \geq 0 \qquad \forall k, r, s, m \tag{13}$$

$$\sum_k f_{k,m}^{rs} = q_m^{rs} \qquad \forall r, s, m \tag{14}$$

Constraints 10, 12, and 14 are simply the flow nonnegativity, distance, and flow conservation constraints of the original problem formulation, respectively. These constraints will hold at the minimum point of the objective function. Define

$$c_k^{rs} = \sum_a t_a \delta_{a,k}^{rs} \tag{15}$$

$$p_{k,m}^{rs} = \sum_a t_a \delta_{a,k}^{rs} + \lambda_{k,m}^{rs*}\left(l_k^{rs} - D_m\right) \tag{16}$$

where c_k^{rs} is the path travel cost, which is the sum of the costs of those links along the path, and $p_{k,m}^{rs}$ can be interpreted to be a composite path cost for the mth class of vehicles. It is the sum of two components: the path travel cost and $\lambda_{k,m}^{rs*}(l_k^{rs} - D_m)$, which is the path out-of-range cost incurred when the path length exceeds the distance limit of the mth class of vehicles. From this definition of $p_{k,m}^{rs}$, one can see that the Lagrangian multiplier $\lambda_{k,m}^{rs}$ stands for the value of distance for the mth type of vehicles, that is, an equivalent travel cost value of the unit travel distance. If a path's total length exceeds a certain type of vehicle's distance limit, the out-of-range cost is set positive so that the composite cost is greater than or equal

to the minimum-path cost and the chance that this path will be chosen decreases to 0 (i.e., Equations 8 to 10).

Constraints 11 to 13 make sure that any path's composite cost is at least equal to its travel cost. Specifically, when $(D_m - l_k^{rs}) f_{k,m}^{rs} > 0$, that is, the kth path connecting O-D pair r–s is used by the mth class of vehicles and its length is less than the distance limit of the mth class of vehicles, $\lambda_{k,m}^{rs}$ must equal 0 for Equation 11 to hold. In this case, the composite path cost is equal to the path travel cost; otherwise, $\lambda_{k,m}^{rs} > 0$ and additional cost is incurred.

Substitution of Equation 16 into Equations 8 and 9 provides

$$f_{k,m}^{rs*}\left(p_{k,m}^{rs} - u_m^{rs*}\right) = 0 \qquad \forall k, r, s, m \tag{17}$$

$$p_{k,m}^{rs} - u_m^{rs*} \geq 0 \qquad \forall k, r, s, m \tag{18}$$

These two conditions state the user equilibrium principle under the distance constraint and hold for each path between any O-D pair in the network. For a given path, say, path k connecting r and s, the conditions hold for two possible combinations of path flow and path cost. Either the flow of the mth class of vehicles on that path is 0 (that is, $f_{k,m}^{rs}$ is equal to 0 and Equation 17 holds; in this case, the composite travel cost for the mth class of vehicles on that path, $p_{k,m}^{rs}$, must be greater than or equal to u_m^{rs}, as required by Equation 18) or the flow of the mth class of vehicles on that path is positive, in which case $p_{k,m}^{rs}$ is equal to u_m^{rs} and both Equation 17 and Equation 18 hold. In any case, the value of u_m^{rs} is less than or equal to the composite travel cost of the mth class of vehicles on all paths connecting r and s; that is, u_m^{rs} is the minimum composite travel cost of the mth class of vehicles traveling from r to s.

It is readily proved that Objective Function 1 is convex and that the feasible region defined by Constraints 2 to 4 is convex (28). Therefore, this DCTAP model has a unique solution.

Problem Feasibility

The extra distance constraint in the model introduced above could be infeasible. For some O-D pairs, if none of those paths connecting them satisfies the distance limit of a certain class of vehicles, the travel demand between them for this class of vehicles cannot be assigned to the network and the problem has no feasible solution. Those infeasible O-D pairs, however, can be easily detected. For example, for each O-D pair, the distance of the shortest path is checked, and if the shortest path is longer than a certain class of vehicles' distance limit, then no feasible path for the DCTAP exists between this O-D pair for this class of vehicles.

SOLUTION APPROACH

It was shown that the Frank–Wolfe algorithm can still be applied to the DCTAP, given the problem's convex structure, but with a direction-finding step different from that of the classic TAP. To find a descent direction to the optimization problem in Equations 1 to 4, the Frank–Wolfe algorithm searches the entire feasible region for an auxiliary feasible solution, $\mathbf{y}^n$, such that the direction from $\mathbf{x}^n$ (the current solution at the nth iteration) to $\mathbf{y}^n$ provides a maximum drop in the objective function value (28). This direction can be constructed by solution of the following linearization problem:

$$\min Z^n(\mathbf{y}) = \nabla Z(\mathbf{x}^n) \cdot \mathbf{y}^T = \sum_a t_a(x_a^n) y_a \tag{19}$$

subject to

$$\sum_k g_{k,m}^{rs} = q_m^{rs} \qquad \forall r, s, m \tag{20}$$

$$g_{k,m}^{rs} \geq 0 \qquad \forall k, r, s, m \tag{21}$$

$$\left(D_m - l_k^{rs}\right) g_{k,m}^{rs} \geq 0 \qquad \forall k, r, s, m \tag{22}$$

where $g_{k,m}^{rs}$ is the auxiliary flow rate of the mth class of vehicles on path k connecting O-D pair r–s and y_a is equal to $\sum_{rs} \sum_k \sum_m g_{k,m}^{rs} \delta_{a,k}^{rs}$, $\forall a$, and is the auxiliary link flow rate. Given that x_a^n is the current link flow rate obtained from the last iteration, link travel cost $t_a(x_a^n)$ is constant here. The objective of the program described above is to minimize the total travel cost over a network with fixed link travel costs. The total travel cost spent in the network will be minimized by assignment of all vehicles to the least-cost paths connecting their origins and destinations, whose lengths also satisfy the distance limit. This problem is actually the CSPP described above.

To see that the program described above involves nothing more than a CSPP, the direction-finding step of the Frank–Wolfe method can be derived with the gradient of Objective Function 1 for path flows. The program then becomes

$$\min Z^n(\mathbf{g}) = \nabla_f Z\left[\mathbf{x}(\mathbf{f}^n)\right] \cdot \mathbf{g}^T = \sum_{rs} \sum_k \sum_m c_k^{rs,n} g_{k,m}^{rs} \tag{23}$$

subject to

$$\sum_k g_{k,m}^{rs} = q_m^{rs} \qquad \forall r, s, m \tag{24}$$

$$g_{k,m}^{rs} \geq 0 \qquad \forall k, r, s, m \tag{25}$$

$$\left(D_m - l_k^{rs}\right) g_{k,m}^{rs} \geq 0 \qquad \forall k, r, s, m \tag{26}$$

where $c_k^{rs,n}$ is the travel cost on path k connecting O-D pair r and s at the nth iteration of the algorithm. This program can be decomposed by O-D pair and vehicle class since path travel costs and path lengths are fixed. The resulting subproblem for the mth class of vehicles and O-D pair r–s is minimized by determination of path h, which has the lowest travel cost among all paths connecting origin r and destination s that satisfy the distance constraint of the mth class of vehicles, and then assignment of all the demand for the mth class of vehicles to this path. In other words, $g_{h,m}^{rs}$ is equal to q_m^{rs} if l_h^{rs} is $\leq D_m$ and c_h^{rs} is $\leq c_k^{rs}$ for all k satisfying $l_k^{rs} \leq D_m$, and $g_{k,m}^{rs}$ is equal to 0 for all other paths.

Once the path flow pattern $g_{k,m}^{rs}$ is found, the auxiliary link flow pattern can be calculated; that is, y_a^n is equal to $\sum_{rs} \sum_k \sum_m g_{k,m}^{rs,n} \delta_{a,k}^{rs}$, $\forall a$. The descent direction ($\mathbf{d}^n$) can then be obtained as $\mathbf{y}^n - \mathbf{x}^n$. Once the descent direction is determined, any line search method can be applied to obtain the move size so that the maximum drop of the objective function value is achieved.

Here it has thus been proved that the DCTAP can be solved by use of the Frank–Wolfe procedure and that the direction-finding step of this procedure is equivalent to solution of the CSPP. The specific steps of this approach can be summarized as follows:

Step 0. Feasibility check. For an O-D pair, find the shortest path according to distance. If the length of this path is longer than the range limit of a certain class of vehicle and the demand of this class of vehicle between this O-D pair is positive, then no feasible path exists for this class of vehicle between this O-D pair. Add this O-D pair and corresponding infeasible vehicle classes to Set G. After that is done, check for a feasible path for each O-D pair. If Set G is empty, continue to the next step; otherwise, stop and report Set G.

Step 1. Initialization. For each O-D pair, find the distance-constrained least-cost path for each class of vehicles on the basis of the free-flow travel cost $t_a = t_a(0)$, $\forall a$. Assign all the demand of each class of vehicles between this O-D pair to the corresponding constrained shortest path. This yields $\{x_a^1\}$. Set counter $n := 1$.

Step 2. Update. Calculate a new link cost according to $t_a^n = t_a(x_a^n)$, $\forall a$.

Step 3. Direction finding. For each O-D pair and vehicle class, find the distance-constrained least-cost path on the basis of new link travel cost t_a^n. Assign the demand of each class of vehicles between this O-D pair to corresponding paths. This yields auxiliary flow $\{y_a^n\}$.

Step 4. Line search. Apply any of the interval reduction line search methods, such as the bisection method (which is particularly applicable here), the golden section method, or other applicable method, to find the optimal value of θ—the optimal move size—by determination of the solution to

$$\min_{0 \le \theta \le 1} \sum_a \int_0^{x_a^n + \theta(y_a^n - x_a^n)} t_a(\omega) d\omega$$

Step 5. Move. Set x_a^{n+1} equal to $x_a^n + \theta(y_a^n - x_a^n)$, $\forall a$.

Step 6. Convergence test. If the preset convergence criterion is not met, set $n := n + 1$ and go to Step 1; otherwise, stop and $\{x_a^{n+1}\}$ is the set of equilibrium link flows.

It is readily known that the most computation-intensive step in each iteration of this algorithm is the search for the distance-constrained shortest paths in the direction-finding step. Therefore, an efficient method for detection of the constrained shortest path is utmost to the efficiency of the whole algorithmic procedure. As stated above, the modified label-setting algorithm with a pre-processing phase developed by Dumitrescu and Boland performs well in historical numerical tests (27). Therefore, this method was used to solve the distance-constrained shortest-path problem in the numerical examples.

NUMERICAL EXAMPLES

The purpose of this numerical analysis is threefold: (a) to justify the validity of the model and algorithm, (b) to examine the impact of the distance constraint on network flow patterns, and (c) to evaluate the change of computational costs caused by the distance constraint.

The solution procedure is first applied to a simple network with eight nodes and 10 links, as shown in Figure 1a. The number beside each link is the link length. Nodes 1 and 2 are origins, and Nodes 3 and 4 are destinations. In this small example, the focus is on analysis
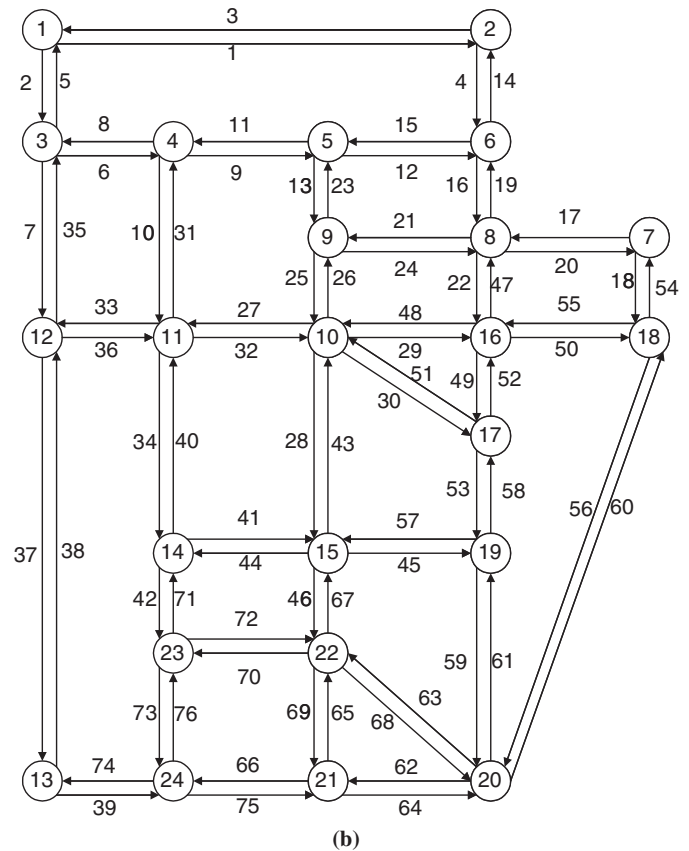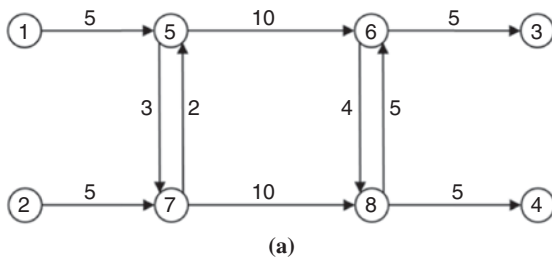

FIGURE 1 Test networks: (a) a small network and (b) Sioux Falls network.

TABLE 1   Link Flow Pattern Under Different Constraint Conditions

| Distance Limit | Link Flow | | | | | | System Total Travel Cost |
	Link 5–6	Link 5–7	Link 6–8	Link 7–5	Link 7–8	Link 8–6	
—	20	5	5	5	20	5	906
27	20	5	5	5	20	5	906
25	20	5	5	5	20	5	906
24	21	9	1	10	19	0	990
23	20	10	0	10	20	0	1,006

NOTE: — = no distance limit.

of how different distance limits affect the routing behavior of vehicles; for simplicity, only a single class of vehicles is considered. The travel demand between each O-D pair (O-D Pairs 1–3, 1–4, 2–3, and 2–4) for this class of vehicles is 10 flow units. Link costs and distances of the connectors (i.e., those links connecting origins or destinations with internal nodes) are assumed to be 0, and all other links have the same cost function, namely, $t_a = 1 + x_a^2$.

The solution procedure described above was run with different values of the distance constraint, and the results are summarized in Table 1. The connectors are not listed in Table 1, since their flow rates will always be 20.

This toy network has, at most, two paths between each O-D pair. Their lengths and costs at equilibrium under different distance limits are shown in Table 2.

Notice from Table 1 that the equilibrium flow patterns under the first three conditions (i.e., no distance limit (D), D equal to 27, and D equal to 25) are the same. The same equilibrium flow patterns can be explained by Table 2, which shows that when no distance limit exists, the second path between O-D Pair 1–3 (1–5–7–8–6–3) and the second path between O-D Pair 2–4 (2–7–5–6–8–4) are not used because their costs are higher than the minimum cost between corresponding O-D pairs. When D is equal to 27, the only path violating the distance limit is the one not used even when no distance limit exists, and a similar situation exists when D is equal to 25. Therefore, the first three conditions have the same flow pattern because the distance constraints in the last two conditions are actually not

binding. When D is equal to 24, only O-D Pair 1–4 has two available paths, and both of them carry flows. Because of this tighter limit, the second path between O-D Pair 2–3 (2–7–8–6–3) is no longer feasible. Traffic flows switch from this path to the first path, which causes the cost of those paths through Link 5–6 to increase (e.g., the cost of Path 1–5–6–3 increases from 401 to 442), whereas the cost of those through Link 7–8 decreases (e.g., the cost of Path 2–7–8–4 decreases from 401 to 362). As the limit becomes tighter, the number of paths used decreases. When D is equal to 23, only one path can be used for each O-D pair. As a result, some links, such as Link 6–8 and Link 8–6, will not be used at all.

The last column of Table 1 shows that the total network travel cost increases as the distance limit gets lower. This is not surprising because when the distance constraint is set to be tighter, the number of feasible paths in the network typically decreases and those remaining feasible paths become more congested. However, this situation is not always the case; when a phenomenon similar to Braess's paradox arises in the distance-constrained traffic network, the total network travel cost may decrease with a tighter constraint. This phenomenon will be discussed in a subsequent paper.

The algorithm was then applied to a larger benchmark network, the Sioux Falls, South Dakota, network, as shown in Figure 1b. The link performance functions and travel demand table used in the application are directly from previous work (29). For simplicity, the free-flow travel time is used as a proxy for the link length for each link.

TABLE 2   Paths Between Each O-D Pair and Their Lengths and Costs

| O-D | Path | Path Length | Path Cost | | | | |
			No Distance Limit	D = 27	D = 25	D = 24	D = 23
1-3	1–5–6–3	20	401	401	401	442	401
	1–5–7–8–6–3	28	453[a]	—[b]	—[b]	—[b]	—[b]
1-4	1–5–7–8–4	23	427	427	427	444	502
	1–5–6–8–4	24	427	427	427	444	—[b]
2-3	2–7–5–6–3	22	427	427	427	543	502
	2–7–8–6–3	25	427	427	427	—[b]	—[b]
2-4	2–7–8–4	20	401	401	401	362	401
	2–7–5–6–8–4	26	453[a]	453[a]	—[b]	—[b]	—[b]

[a]Path is not used because its cost is higher than that of the least-cost path;
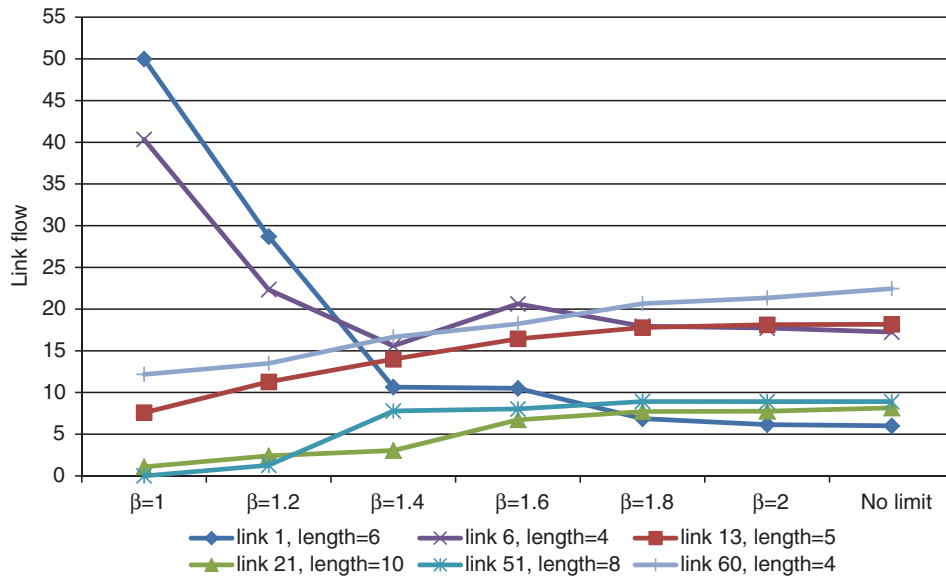[b]Path is not used because its length is out of range.

FIGURE 2   Changes of link flow rates against distance limits.

The link flow patterns under two different cases, with and without the distance limit, are calculated and compared. Without a loss of generality, it was assumed that for each O-D pair, only one class of vehicles exists but that different O-D pairs may contain different classes of vehicles. That is, all the vehicles traveling between the same O-D pair have the same distance range, but vehicles traveling between different O-D pairs may have different range limits. As a convenient numerical setting, the distance limit for vehicles between each O-D pair is set as $\beta \times \min\{l_k^{rs}, \forall k\}, \forall r,s$, where l_k^{rs} is the same as previously defined and β is a parameter. In this example, a β value of 1.2 is used for the case with the distance limit. For a precise comparison, numerical results in both cases are obtained from a running of the Frank–Wolfe algorithm until the convergence criterion based on the changes in flows converges to 1 E−3.

From the result, it was found that the equilibrium flows change significantly (either increase or decrease) on a number of links (Links 1, 3, and 30, to name a few) but change little on other links

(Links 4, 12, and 15, for example). A few example links were randomly selected, and their flow variations with different distance limits were observed (Figure 2).

It is readily seen from Figure 2 that when β begins to increase, the network flows on these links change in a dramatic manner; as β continuously increases, the flow rates change relatively mildly and ultimately converge to their corresponding values with no distance constraints. This is because an increase in β amounts to loss of the distance constraint. However, these changes may not necessarily be monotone.

To test how much computational time was increased in the Sioux Falls network because of the addition of the distance constraint, the number of iterations (ITR), the total computational cost (TCC), and the computational cost per Frank–Wolfe iteration (CCPE) were compared for the basic TAP and DCTAP under different distance limits and convergence criteria. The results are reported in Table 3, which shows that the CCPE of DCTAP is higher than that of TAP

TABLE 3   Computational Costs With or Without Distance Constraints

| Problem Type | β | Computational Cost by Convergence Criteria | | | | | | | | | | |
| | | 1 E−1 | | | 1 E−2 | | | 1 E−3 | | | 1 E−4 | | |
		ITR	TCC (s)	CCPE (s)	ITR	TCC (s)	CCPE (s)	ITR	TCC (s)	CCPE (s)	ITR	TCC (s)	CCPE (s)
TAP	1	1	0.084	0.084	5	0.437	0.087	39	3.119	0.080	225	18.214	0.081
DCTAP													
	1	2	0.478	0.239	2	0.430	0.215	3	0.612	0.204	3	0.627	0.209
	1.2	1	0.222	0.222	3	0.497	0.166	3	0.539	0.180	3	0.503	0.168
	1.4	1	0.190	0.190	4	0.658	0.165	10	1.617	0.162	16	2.486	0.155
	1.6	1	0.166	0.166	3	0.358	0.119	6	0.751	0.125	8	0.851	0.106
	1.8	1	0.154	0.154	3	0.337	0.112	15	1.403	0.094	97	8.393	0.087
	2	1	0.147	0.147	3	0.319	0.106	24	2.243	0.093	179	14.745	0.082
	3	1	0.103	0.103	4	0.395	0.099	40	3.346	0.084	222	17.821	0.080

and that the tighter the constraint is (i.e., the smaller that the value of β is), the higher is the value of CCPE. For example, when the convergence criterion is 1 E−4 and $\beta = 1$, the CCPE of DCTAP is about 2.6 times that of TAP. However, the TCC of DCTAP under this condition is much lower than that of TAP, with the value for the latter being almost 11.5 times that for the former. This result is because DCTAP requires only three iterations, whereas TAP needs 225 to converge. Although solution of the distance-constrained shortest-path problem costs more time than solution of the shortest-path problem, when the constraint is tighter, a smaller number of paths are eligible for assignment of traffic and fewer iterations are thus needed to achieve the equilibrium condition. When the distance constraint is loose, for example, when $\beta \geq 3$, both the TCC and CCPE of DCTAP are close to those of TAP.

CONCLUSIONS AND RESEARCH EXTENSIONS

A new TAP with the path-based distance constraint is formulated, solved, and numerically analyzed. This problem represents a simplified case of traffic networks that carry electric vehicles with various distance limits.

This paper shows that the path cost structure associated with this problem at the equilibrium point is different from that of the basic TAP. It contains a special path-specific out-of-range cost term that makes this problem not as easily solved as the classic problem. The well-known and widely used Frank–Wolfe procedure is adopted to solve this problem. The feasible direction-finding step, however, involves a CSPP, which is essentially NP-complete, but can be solved relatively efficiently by a modified label-setting algorithm with preprocessing. The proposed algorithm is easy to understand and implement. The application of the algorithm for a couple of example problems justifies that the validity and the applicability of the solution procedure to general networks with resource constraints. The numerical results show the impact of the extra resource constraints on network flows and the relationship between the network flow evolution and the tightness of the resource constraints.

The DCTAP model provides a basic mathematical tool to study traffic networks with electric vehicles. This research can be extended along a few different directions. On the modeling side, more complex network equilibrium models with vehicles of different types of path constraints and cost structures are of particular interest. An example is extension of the model to accommodate emerging hybrid vehicles that are not subject to the distance limit but that have different operational costs before and after their battery range limits are exceeded.

The distance constraint has an influence on urban travels and activities deeper and wider than that shown in this paper; in its simplest case, it changes not only individual route choice behavior but also, at least, destination choice and mode choice behaviors. For promotion of a more realistic modeling tool, the authors plan to incorporate the mode choice and destination choice components into the model.

An additional benefit of this combined model is that the possibility that it will be infeasible because of the distance constraint is lower. Incorporation of these extra travel choice behaviors will further increase the model's structural complexity but will allow analysis of more sophisticated and realistic travel demand and network flow patterns.

Consideration of the destination choice behavior subject to the availability of charging stations and the charging price at destina-

tions, for example, is one important piece of the authors' extensive research activities at present, but many other pieces need to be addressed in future endeavors. On the computational side, the plan is to conduct experiments on large-scale networks of realistic size to gain information about how network flow pattern shifts and how much the computational cost increases with the addition of the distance constraint. However, the implementation of the Frank–Wolfe algorithm to solve DCTAP provides only a sublinear convergence rate, and its poor convergence performance in the neighborhood of the optimal solution is notorious; to find highly precise solutions in large-scale implementations, a solution algorithm of the Newton or quasi-Newton type is anticipated.

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