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Linear Programming Formulation for Strategic Dynamic Traffic Assignment

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Abstract This work introduces a novel formulation of system optimal dynamic traffic assignment that captures strategic route choice in users under demand uncertainty. We define strategic route choice to be that users choose a path prior to knowing the true travel demand which will be experienced (therefore users consider the full set of possible demand scenarios). The problem is formulated based on previous work by Ziliaskopoulos (Transp Sci 34(1):37–49, 2000). The resulting novel formulation requires substantial enhancement to account for path-based flows and scenario-based stochastic demands. Further, a numerical demonstration is presented on a network with different demand loading profiles. Finally, model complexity, implications on scalability and future research directions are discussed.

Keywords Dynamic traffic assignment · Demand uncertainty · Linear programming

1 Introduction

With the continuing progress of deployable Dynamic Traffic Assignment (DTA) approaches, researchers are asking fundamentally new questions with regard to model realism and predictability. The core behavioral assumption within many DTA models is that equilibrium route choice exists. This assumption provides substantial descriptive capabilities but also a potential weakness due to the rarity of observed traffic patterns to be in an equilibrium state. Furthermore, this observation is often employed as the key criticism of DTA approaches to an even greater degree than to traditional static techniques due to the ability of the dynamic models to generally represent

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traffic in a more realistic and comparable manner to observable conditions. Numerous approaches have emerged to address this criticism, often referred to as disequilibrium, stochastic, day-to-day, or transient modeling.

An approach that could potentially enhance the behavioral capabilities of DTA models and mitigate the aforementioned criticisms regarding observable equilibria is the concept of ***strategic assignment***, first introduced by Marcotte and Nguyen (1998). In strategic assignment approaches, the resulting model does not attempt to optimize path (or link) flow directly, but rather to discern strategies that are applied following the realization of some uncertain variable. Often, the goal of the new strategic approach is to equilibrate an expected condition as opposed to a deterministic cost equilibration. For the current analysis, we focus on a priori approaches to develop a model that provides superior insights into how traffic volatility emerges in the presence of uncertainty.

Specifically, this paper expands the linear programming System Optimal Dynamic Traffic Assignment (SODTA) modeling framework developed by Ziliaskopoulos (2000), which embeds the cell transmission model (Daganzo 1994) for traffic propagation. While system optimal conditions are modeled (i.e., marginal cost equilibration) due to the specific capabilities of the LP approach, substantial analytical tools become utilizable, such as stochastic linear programming. The novelty of the presented approach is the development of a two-stage stochastic variation where trip demand is uncertain but represented as a random variable that gives rise to multiple potential future scenarios (characterizing different days none of which are in equilibrium/optimality when viewed myopically). In the first stage, routing strategies (i.e., flow proportions) are developed to minimize expected system cost and then employed in the second stage after the trip demands are realized to produce scenario-dependent dynamic flows and densities.

This work is organized in the following manner. Section 2 discusses previous literature on the topics of dynamic traffic assignment, linear programming formulations of DTA, and efforts at modeling daily volatility of traffic flow, followed by an introduction to the idea of strategic assignment. Section 3 briefly reviews the LP DTA formulation from Ziliaskopoulos for completeness, then Section 4 discusses the model properties, followed by Section 5 containing the problem formulation for the strategic SO (StrSO) DTA problem. Section 6 examines computational results and realized flow variability and this paper concludes in Section 7 with a discussion of future research directions.

2 Literature Review

A great number of advancements in the field of Dynamic Traffic Assignment (DTA) have been proposed since the ground-breaking work of Merchant and Nemhauser (1978a, b). The appeal of DTA lies in its ability to capture time-varying flows and thereby achieve superior representations of vehicular traffic. The addition of temporal phenomena, though, substantially complicates the mathematical representation of traffic assignment. For example, analytical methods—usually based on mathematical programming, optimal control theory, or variational inequality approaches—may be limited by their dependence on link performance functions, the holding of traffic, or the inability to scale up to realistic sized problems. However, heuristic approaches, often simulation-based, lack the capacity to guarantee global optimality or to offer significant insight on the problem. See Peeta and Ziliaskopoulos (2001) for a more comprehensive review of keystone DTA literature prior to the year 2000 and discussion on various proposals for overcoming the aforementioned shortcomings.

2.1 LP Formulations

Linear programming formulations for SODTA were first introduced with the independent works of Ziliaskopoulos (2000) and Carey and Subrahmanian (2000). Carey and Subrahmanian (2000) use link performance functions to characterize the relationship between link flow and link cost. While there are certain drawbacks in using link-based performance functions, critical insights can be obtained into the core problem.

This work expands the linear programming SODTA model presented by Ziliaskopoulos (2000), which will be reviewed in Section 3. Novel at the time of its conception, this model draws on Daganzo's cell transmission model (CTM) (1994, 1995) to present a simple formulation of traffic flow that captures flow variability inside the link while avoiding the some of the drawbacks associated with link performance functions. In addition, a number of works refining the original model have been produced. Li et al. (1999) expand upon this LP formulation of SODTA to allow for multi-origins and multi-destinations, while preserving first in first out (FIFO) requirements. Li et al. (2003) propose a decomposition algorithm for the LP SODTA that allows it to be solved on more meaningful sized networks. Waller and Ziliaskopoulos (2006) extend the deterministic LP to account for stochastic demands using a chance-constrained stochastic program and provide solution techniques.

2.2 Day-to-Day Traffic Flow Volatility

Examining the effects of daily volatility in traffic flow is a problem that can be considered from multiple aspects. For example, one source of daily variation may be a result of supply side reductions in capacity, like that resulting from traffic incidents or adverse weather conditions (Asakura and Kashiwadani 1991; Clark and Watling 2005). However, daily variation can also imply an uncertain travel demand which is the focus of this work.

Traditional equilibrium models are, as the name suggests, dependent upon the idea of equilibrium—that is, a consistent state of the network that will remain under some rational set of behaviors. Refer to Watling and Hazelton (2003) for an in-depth discussion on the definition of equilibrium. These authors note a common criticism of traffic assignment models (dynamic included) that questions the existence of an observable equilibrium, a critique that this work seeks to address. Building on this issue, and the field of behavioral dynamics, alternative schools of thought claim that decisions are affected by learning from previous days, leading to potentially unstable conditions; users adapt on a day-to-day basis, and their adapting mechanism may undermine the notions of a stable equilibrium (Hamdouch et al. 2004). Understanding this new concept of dis-equilibrium has become an important area of research, in spite of the mathematical and computational complexity of the problem.

Closely related to the issue of day-to-day uncertainty in traffic flow, is the issue of uncertainty in the very behavior exhibited by the decision-makers being modeled in the system. This uncertainty goes beyond just that of user perception in travel time (as addressed with stochastic user equilibrium models, see Daganzo and Sheffi (1977), Sheffi and Powell (1982), Maher and Hughes (1997)). Traditionally, traffic assignment models have built upon the assumption that people seek to minimize their own travel time, but a number of works have explored diffing behavioral assumptions.

Horowitz (1984) examines the stability of stochastic equilibrium in a two-link network by analyzing different mechanisms of route choice over time. He shows that even when equilibrium solutions are unique, link flow values may converge to their equilibrium values, oscillate about equilibrium perpetually, or converge to a solution not consistent with equilibrium conditions. Cantarella and Cascetta (1995) undertake research on interperiodic demand modeling from both a deterministic and stochastic process approach. Rather than focusing on the concept of equilibrium, this work focuses on fixed point attractors. Additionally this work discusses the conditions for stability of both equilibrium and dynamic processes, and the relationship between the two. Watling (1999) extends Horowitz's two-link example to a general network setting, further clarifying the distinction between stability in discrete vs. continuous time, and that between deterministic and stochastic processes. A dynamical adjustment process is proposed for analyzing the stability of a general asymmetric stochastic equilibrium assignment problem in discrete time. Watling and Hazelton (2003) further extend the concept of dynamic learning route choice and examine the properties of deterministic dynamic systems under perturbation, and the implications of day to day route choice adjustments on the stability of equilibria. The authors further note the importance of the behavioral mechanism in modeling day-to-day fluctuations, and emphasize the need for more analytical techniques over simulation based techniques for their potential to offer greater insight.

Rather than focusing on a priori path choice decision making, an alternative method proposes that people may adapt their route choice based on primary, real-time experience. Referred to in the literature as adaptive routing (or routing with recourse), it is assumed that users may gain information and correspondingly change their behavior *en route*. There is an abundance of research in the literature dealing with travel-time adaptive shortest paths (Andreatta and Romeo 1988; Psaraftis and Tsitsiklis 1993; Polychronopoulos and Tsitsiklis 1996; Miller-Hooks and Mahmassani 2000; Waller and Ziliaskopoulos 2002; Fan and Nie 2006; Gao and Chabini 2006; Boyles and Waller 2007). However, extensions of this type of user-level behavior to a system equilibrium are much scarcer. Unnikrishnan and Waller (2009) extend the online shortest path behavior described in Waller and Ziliaskopoulos (2002) to a user equilibrium framework using a convex math programming formulation.

2.3 Strategic Assignment

Strategic assignment refers to assumptions about the behavioral aspect of the model. In strategic assignment, users may know of a range of possible system states, and adopt a route which they consider best over this range. In this way, a strategy is a plan that encompasses all possible outcomes and defines an action for each possible scenario. The idea of strategic decision-making was first introduced as applied to transit users (Chriqui and Robillard 1975). Marcotte and Nguyen (1998) formulated a traffic equilibrium approach for transit and capacitated networks using hyperpaths that yielded valuable insights. Later this approach was further developed by Marcotte et al. (2004), and extended to traffic networks with rigid finite capacities. This work introduced the concept of strategies, where at each node, users are assigned a set of sub-paths and an order of preferences. Instead of an explicit cost function, this work employs the assumption of a rigid finite capacity. The problem was formulated as a variational inequality, and five solution algorithms were compared. Hamdouch et al. (2004) expanded this method to a dynamic formulation in which again users follow strategies based on preferences, while meeting first in first out requirements.

The problem examined in this research differs from that in the literature in that it incorporates the concept of a strategic approach to equilibrium within a linear programming framework, and further captures the finer grain resolution phenomena observable through the use of CTM. While the complexity of the formulation presented here preempts it from being scaled up to larger networks, the linear programming framework allows for the implementation of well-established decomposition methods, which are discussed.

3 Overview of the SO-DTA Model

For the purposes of continuity and comparison, Ziliaskopoulos' (2000) original model formulation is briefly recounted here. Table 1 contains a truncated explanation of the notation used in this paper (Table 1 contains the notation for the SO-DTA model), while Table 2 contains the expanded version of the notation.

The following SO-DTA model is formulated as a linear program with the objective of minimizing total travel time. This model consistently propagates traffics through a

Table 1 Notation for the SO-DTA model

C	Set of all cells
E	Set of all cell connectors
C_R, C_S	Set of origin (source) cells and destination (sink) cells, respectively
t	Time period index
T	Set of discrete time intervals, i.e., $T = \{\pi, 2\ \pi, 3\ \pi, \ldots, \lvert T\rvert\ \pi\}$, and with no loss of generality, assume that $\pi=1$
d_i^t	Demand at cell i at time t
x_i^t	Number of vehicles at time interval t on cell i
y_{ij}^t	Flow from cell i to cell j at time interval t
N_i^t	Maximum number of vehicles that can be present in cell i at time interval t
Q_i^t	Maximum number of vehicles that can flow into or out of cell i during time interval t
$P(i)$	Set of predecessor cells to cell i
$S(i)$	Set of successor cells to i

single destination network, with expressions (2)–(9) as linear constraints (Ziliaskopoulos 2000):

$$Minimize \sum_{\forall t \in T} \sum_{\forall i \in C \backslash C_s} x_i^t \tag{1}$$

s. *t*.

$$x_i^t - x_i^{t-1} - \sum_{k \in P(i)} y_{ki}^{t-1,} + \sum_{j \in S(i)} y_{ij}^{t-1} = 0, \quad \forall i \in C \backslash \{C_R, C_S\}, \forall t \in T, \tag{2}$$

$$\sum_{\forall j \in S(i)} y_{ij}^t - x_i^{t,} \leq 0, \quad \forall i \in C, \forall t \in T \tag{3}$$

$$\sum_{\forall i \in P(j)} y_{ij}^t + x_j^t \leq N_j^t, \quad \forall j \in C \backslash \{C_R, C_S\}, \forall t \in T \tag{4}$$

$$\sum_{\forall i \in P(j)} y_{ij}^t \leq Q_j^t, \quad \forall j \in C \backslash C_R, \forall t \in T, \tag{5}$$

$$\sum_{\forall j \in S(i)} y_{ij}^t \leq Q_i^t, \quad \forall i \in C \backslash C_S, \forall t \in T \tag{6}$$

$$x_i^t - x_i^{t-1} + y_{ij}^{t-1} = d_i^{t-1} \quad \forall j \in S(i), \forall i \in C, \forall t \in T \tag{7}$$

$$x_i^0 = 0, y_{ij}^0 = 0 \quad \forall i \in C, \forall (i,j) \in E, t \in T \tag{8}$$

$$x_i^t \geq 0, y_{ij}^t \geq 0, \quad \forall i \in C, \forall (i,j) \in E, t \in T \tag{9}$$

Constraints (2) enforces conservation of flow, while constraint (3) ensures that the amount of flow that can be sent is restricted by the current cell density. Constraint (4) addresses the capacity of the cell. Furthermore, note that the capacity of origin and destination cells is assumed to be infinity in order to allow proper loading of the network. Constraints (6) and (7) limit the total inflow and outflow of a cell, while constraint (7) loads demands onto origin cells. Constraints (8) and (9) represent initial conditions and non-negativity constraints.

The formulation presented above is cell-centric; the path flows are not directly computed or represented in the formulation. Because of this, extracting path-based information is not trivial, and because the route choice element is critical in the focus of this paper, the method developed needs to explicitly account for path flows. In the following section, we elaborate on this formulation to account for time of departure-based path flows, within a scenario-based stochastic demand framework.

4 Problem Formulation

This paper addresses the impact of demand uncertainty on traditional equilibrium planning models. Instead of assuming a deterministic demand value and then determining the optimal paths for travelers, this model identifies the optimal *proportion* of flow on each path (for each possible departure time) to minimize *expected* total system travel time across a range of specified discrete demand scenarios. Hence, instead of three different demand scenarios resulting in three different optimal solutions, the model will output a single optimal value; however, no demand scenario is an optimal solution *in and of itself*. The optimality results from a greater strategy prevailing across all potential demand scenarios and may be interpreted as an explanation for the randomness that can be observed in traffic flow.

The focus of this section is two-fold: to formulate the scenario-based optimal path proportion problem as a linear program, and to re-interpret the optimal path proportions to represent the probability distribution guiding the path choice of each individual belonging to that OD.

The formulation of the LP follows that of Ziliaskopoulos (2000) for the SODTA problem, but increases in complexity due to the need to explicitly track paths, and the scenario-based demand stochasticity. In order to estimate the optimal path proportions in this problem, it is necessary to explicitly separate flows based on both departure time and path; conceptually speaking, the problem can be modeled as a set of separate single origin, single destination DTA problems, one for each path and departure time combination, which share cell and flow capacities at each time step. This observation is important, as it serves as a starting point for the potential use of decomposition methods similar to those used by Li et al. (2003).

The LP formulation presented in this paper aims to find the set of optimal path proportions over all demand scenarios, for each OD pair and departure time. Given these path proportions, scenario-specific demands are loaded onto the network using a CTM-based LP formulation. An expanded explanation of the notation used in the remainder of this paper is shown in Table 2.

The re-interpretation of the path proportions is natural; the expected number of people to follow each path will be equal to the optimal path flows. However, as the flows are being randomly sampled from this distribution, any specific sample taken from this distribution will in general not represent an optimal solution. The variability in the path flows resulting from this sampling approach is a natural way of representing the day to day variability in flows. This variability in path flows can be extended to link flows, allowing us to measure the reliability and/or volatility of links in the network.

The objective of the problem formulated in this section is to find, for each origin–destination (OD) pair, the proportion of flow assigned to each feasible path at each departure time such that the total expected system travel time is minimized. This assumes that the demand for each OD can be any of a set of discrete values with a specific probability. The problem is formulated and solved as a linear program where path proportions, density and flow variables are chosen so as minimize total expected system travel time. The approach is similar to that of Ziliaskopoulos (2000) in that it represents the cell transmission model as a linear program. The objective function for the strategic SO DTA formulation is the expected total system travel time, where

Table 2 Expanded notation table

C	Set of all cells
E	Set of all cell connectors
C_R, C_S	Set of origin (source) cells and destination (sink) cells, respectively
τ	Departure time index
t	Time period index
T	Set of discrete time intervals, i.e., $T = \{\pi, 2\ \pi, 3\ \pi, \ldots, \lvert T\rvert\ \pi\}$, and with no loss of generality, assume that $\pi=1$
Ξ	Set of demand scenarios.
ξ	Demand scenario index.
$d_\tau^{rs,\xi}$	Demand between OD pair (r, s) at departure time τ in demand scenario ξ
RS	Set of all OD pairs $\{(r,s): r\in C_R, s\in C_S, \sum_{\tau=0}^{T} d_\tau^{rs}>0\}$
$\Phi(rs)$	Set of all paths, $\phi_1^{rs}\ldots,\phi_i^{rs}$ connecting OD pair (r, s)
$x_{i,\phi,\tau}^{t,rs,\xi}$	Number of vehicles at time interval t on cell i which departed at time τ, following path ϕ between origin r and destination s in demand scenario ξ
$x_i^{t,\xi}$	Number of vehicles contained in cell i at time interval t in demand scenario ξ
$y_{ij,\phi,\tau}^{t,rs,\xi}$	Flow from cell i to cell j at time interval t for OD pair rs with departure time τ and travelling along path ϕ in demand scenario ξ
$\omega_i^{t,\xi}$	Total flow into cell i at time t in demand scenario ξ
$\psi_i^{t,\xi}$	Total flow out of cell i at time t in demand scenario ξ
N_i^t	Maximum number of vehicles that can be present in cell i at time interval t
Q_i^t	Maximum number of vehicles that can flow into or out of cell i during time interval t
$P(i)$	Set of predecessor cells to cell i
$S(i)$	Set of successor cells to i
δ_ϕ^{ij}	Indicator equal to 1 if cell connector (i, j) is included along path ϕ, and 0 otherwise
$p_{\phi\tau}^{rs}$	Probability of using path ϕ at departure time τ for OD pair rs
p^ξ	Probability associated with demand scenario $\xi\{p:\sum_\xi p^i=1\}$ (for discrete demand scenarios)

because of the underlying cell transmission model, the total system travel time is the number of vehicles in each cell over all time period. This total system travel time for each individual scenario is multiplied by the probability that scenario, and the sum of all demand scenarios is the function to be minimized by the LP. Note that when each demand scenario has equal probability, this term will fall out of the objective function. The formulation is shown below.

$$Minimize \sum_{\forall\xi\in\Xi}\sum_{\forall t\in T}\sum_{\forall i\in C\setminus C_S} p^\xi x_i^{t\xi} \tag{10}$$

$s.\ t.$

$$x_{i,\phi,\tau}^{t,rs,\xi} - x_{i,\phi,\tau}^{t-1,rs,\xi} - \sum_{k\in P(i)} \delta_\phi^{ki} y_{ki,\phi,\tau}^{t-1,rs,\xi} + \sum_{j\in S(i)} \delta_\phi^{ij} y_{ij,\phi,\tau}^{t-1,rs,\xi} = 0, \quad \forall i\in C\setminus\{C_R, C_S\}, \forall t\in T, \forall\phi\in\Phi(rs), \forall rs\in RS, \forall\xi\in\Xi \tag{11}$$

$$x_{i,\phi,\tau}^{t,rs,\xi} - x_{i,\phi,\tau}^{t-1,rs,\xi} - \sum_{k\in P(i)} \delta_{\phi}^{ki} y_{ki,\phi,\tau}^{t-1,rs,\xi} = 0 \quad \forall\ s \in C_S\ ,\ \forall\ t \in T\ , \quad \forall\phi\in\Phi(rs), \forall rs\in RS, \forall\xi\in\Xi \tag{12}$$

$$x_{r,\phi,\tau}^{t,rs,\xi} - x_{r,\phi,\tau}^{t-1,rs,\xi} + \sum_{j\in S(r)} \delta_{\phi}^{rj} y_{rj,\phi,\tau}^{t-1,rs,\xi} = p_{\phi\tau}^{rs} d_{t-1}^{rs,\xi} \quad \forall\ r \in C_R\ ,\ \forall\ t \in T\ , \quad \forall\phi\in\Phi(rs), rs\in RS, \forall\xi\in\Xi \tag{13}$$

$$\sum_{j\in S(i)} \delta_{\phi}^{ij} y_{ij,\phi,\tau}^{t,rs,\xi} - x_{i,\phi,\tau}^{t,rs,\xi} \le 0, \quad \forall i\in C\setminus C_S, \forall t\in T, \forall\xi\in\Xi \quad \forall\phi\in\Phi(rs), \forall(rs)\in RS, \forall\xi\in\Xi \tag{14}$$

$$\omega_i^{t,\xi} + x_i^{t,\xi} \le N_i^t, \quad \forall i\in C\setminus\{C_R, C_S\}, \forall t\in T, \forall\xi\in\Xi \tag{15}$$

$$\omega_i^{t,\xi} \le Q_i^t, \quad \forall i\in C\setminus C_R, \forall t\in T, \forall\xi\in\Xi \tag{16}$$

$$\psi_i^{t,\xi} \le Q_i^t, \quad \forall i\in C\setminus C_S, \forall t\in T, \forall\xi\in\Xi \tag{17}$$

$$\sum_{\forall\phi\in\Phi(r,s)} p_{\phi t}^{rs} = 1 \quad \forall rs\in RS, \forall t\in T, \forall\xi\in\Xi \tag{18}$$

$$x_i^{t,\xi} = \sum_{RS} \sum_{\phi\in\Phi} \sum_{\tau=0}^{t} x_{i,\phi,\tau}^{t,rs,\xi} \quad \forall i\in C, \forall t\in T, \forall\xi\in\Xi \tag{19}$$

$$\omega_i^{t,\xi} = \sum_{RS} \sum_{\phi\in\Phi} \sum_{\tau=0}^{t-1} \sum_{k\in P(i)} \delta_{\phi}^{ki} y_{ki,\phi,\tau}^{t,rs,\xi} \quad \forall i\in C, \forall t\in T, \forall\xi\in\Xi \tag{20}$$

$$\psi_i^{t,\xi} = \sum_{RS} \sum_{\phi\in\Phi} \sum_{\tau=0}^{t-1} \sum_{j\in S(i)} \delta_{\phi}^{ij} y_{ij,\phi,\tau}^{t,rs,\xi} \quad \forall i\in C, \forall t\in T, \forall\xi\in\Xi \tag{21}$$

$$x_{i,\phi,\tau}^{0,rs,\xi} = 0, \quad \forall i\in C\setminus\{C_R, C_S\}, \tau\in T, \forall rs \in RS\ ,\ \forall\ \phi \in \Phi\ (rs)\ ,\ \forall\ \xi \in \Xi \tag{22}$$

$$y_{ij,\phi,\tau}^{0,rs,\xi} = 0 \quad \forall (i,j) \in E, \tau \in T, \forall rs \in RS, \forall \phi \in \Phi(rs), \forall \xi \in \Xi \tag{23}$$

$$x_{i,\phi,\tau}^{t,rs,\xi} \geq 0, \quad \forall i \in C \backslash \{C_R, C_S\}, \tau \in T, \forall rs \in RS, \forall \phi \in \Phi(rs), \forall \xi \in \Xi \tag{24}$$

$$y_{ij,\phi,\tau}^{t,rs,\xi} \geq 0 \quad \forall (i,j) \in E, \tau \in T, \forall rs \in RS, \forall \phi \in \Phi(rs), \forall \xi \in \Xi \tag{25}$$

$$p_{\phi,t}^{rs} \geq 0 \quad \forall t \in T, \forall \phi \in \Phi(rs), \forall rs \in RS \tag{26}$$

Intuitively, the formulation given above can be interpreted as a set of SODTA LPs, each one corresponding to a particular demand scenario, where the path proportion variables that assign OD demand to path flows is fixed across all demands. This property of the formulation is the basis for the discussion on decomposition methods provided later in the paper.

The objective function minimizes the occupancy of all cells over all time periods, which is equivalent to minimizing the total expected system travel time. Constraints (11), (12) and (13) represent the disaggregate flow conservation constraints for regular cells, destinations, and origins respectively. It is worth noting that the path proportion variables enter the problem as part of the conservation of flow constraints for the origin cells (13).

Constraint (14) limits outgoing flow based on the current cell capacity, that is, the amount of flow leaving a cell cannot exceed the number of units currently at the cell. Constraint (15) enforces cell capacity, limiting the total number of units at each cell plus all incoming flow at each time period to be at most the cell capacity, N_i^t. Constraints (16) and (17) enforce limits on the amount of flow a cell can send or receive. Constraint (18) ensures that the set of path proportions satisfy the second axiom of probability, i.e., the sum of the proportions corresponding to all paths sum to 1. Expressions (19), (20) and (21) provide the aggregated measures of flow and density based on their disaggregate equivalents. Constraints (22) and (23) give initial conditions for the problem, and finally constraints (24), (25), and (26) provide non-negativity constraints for the decision variables.

The approach taken in this paper differs from the SODTA formulation of Ziliaskopoulos (2000) in that path flows are solved for explicitly. While approaches can be developed to estimate path flows from cell flows, such approaches are not expected to be simple or scalable based on complexity of the equivalent static problem.

5 Numerical Results

In this section we focus on the implementation and performance of the path-based SODTA algorithm, and subsequently analyze the effects of using path proportions to generate path flows on link flow and link cost variability.

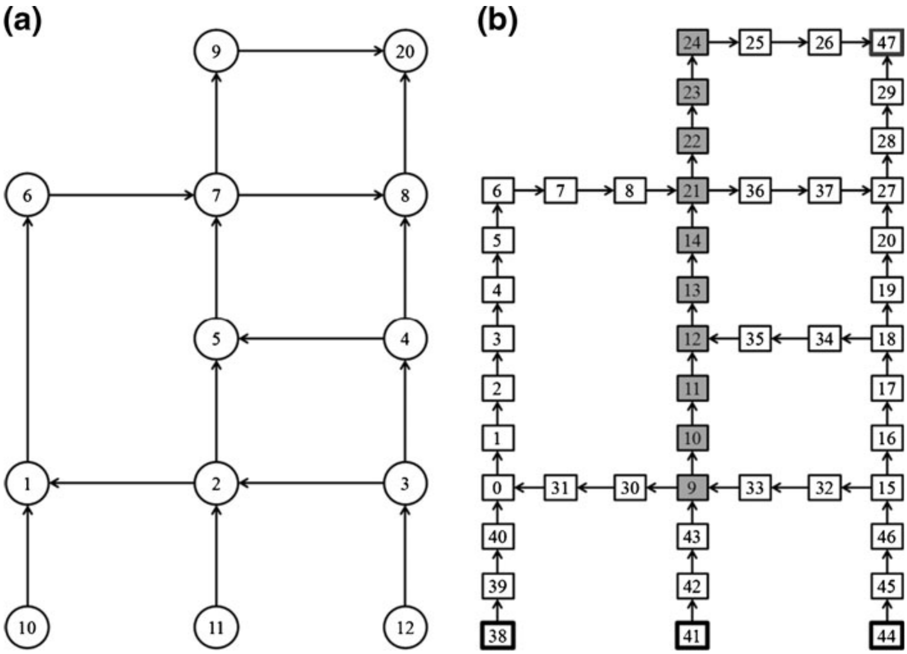

Fig. 1 **a** Network used for numerical results **b** Equivalent cell network with higher capacity cells highlighted

5.1 Description of Test Network

The network considered for numerical testing of the problem and algorithm presented in this paper is a 12 node network with three origin nodes (10, 11 and 12), and one destination node (20), shown in Fig. 1(a). It is assumed that the central corridor, i.e., the node sequence 2–5–7–9, represents an arterial road, while the rest of the links in the networks represent lower capacity roads. The associated cell network is shown in Fig. 1(b). The cell parameters are shown in Table 3.

The characteristics of the resulting cell network are shown in Table 4. For this network, two demand situations were considered, a lightly loaded network and a heavily loaded network. Each demand scenario consisted three levels of demand for three OD pairs that loaded the network over 8 departure time periods ($t=1, \ldots, 8$). Traditionally, this situation would need to be run for the 6 demand scenarios separately, or run twice using an average value for the lightly loaded and heavily loaded network respectively. Using StrSODTA, we again demonstrate the model using two computational runs, but the optimal solution accounts for *all* demand

Table 3 Cell properties

Cells	Max density N_i	Max flow Q_i	Length
9, 10, 11, 12, 13, 14, 21, 22, 23, 24	20	10	450
All other cells	10	3	225

Table 4 Lightly loaded network demands for each time period

Origin	Dest.	Scenario 1 t=1,..,8	Scenario 2 t=1,..,8	Scenario 3 t=1,..,8
38	47	20,20,20,20,10,10,10,10,10	10,10,10,10,10,10,10,10	25,25,25,5,5,5,5,5
41	47	4,4,4,4,4,4,4,4	8,8,8,8,8,8,8,8	12,12,12,12,12,12,12,12
44	47	5,5,5,25,25,5,5,5	25,25,5,5,5,5,5,5	5,5,5,5,5,5,25,25

Table 5 Heavily loaded network demands for each time period

Origin	Dest.	Scenario 1 t=1,..,8	Scenario 2 t=1,..,8	Scenario 3 t=1,..,8
38	47	40,40,40,40,20,20,20,20	20,20,20,20,20,20,20,20	50,50,50,15,15,15,15,15
41	47	10,10,10,10,10,10,10,10	20,20,20,20,20,20,20,20	30,30,30,30,30,30,30,30
44	47	10,10,10,30,30,10,10,10	30,30,10,10,10,10,10,10	10,10,10,10,10,10,30,30

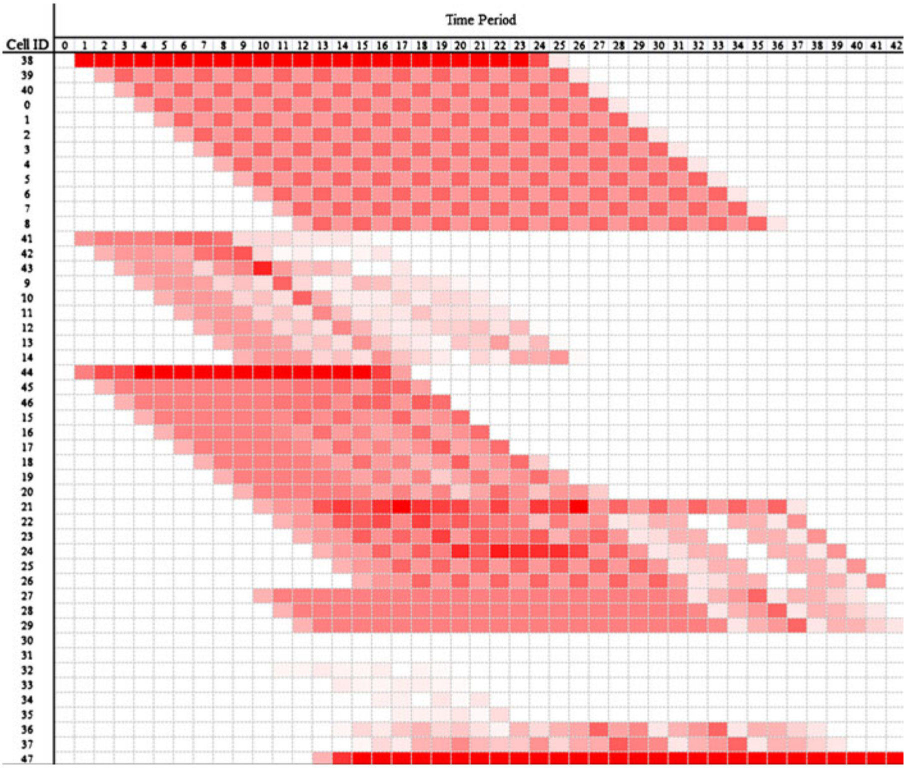

Fig. 2 Lightly loaded network, demand scenario 1

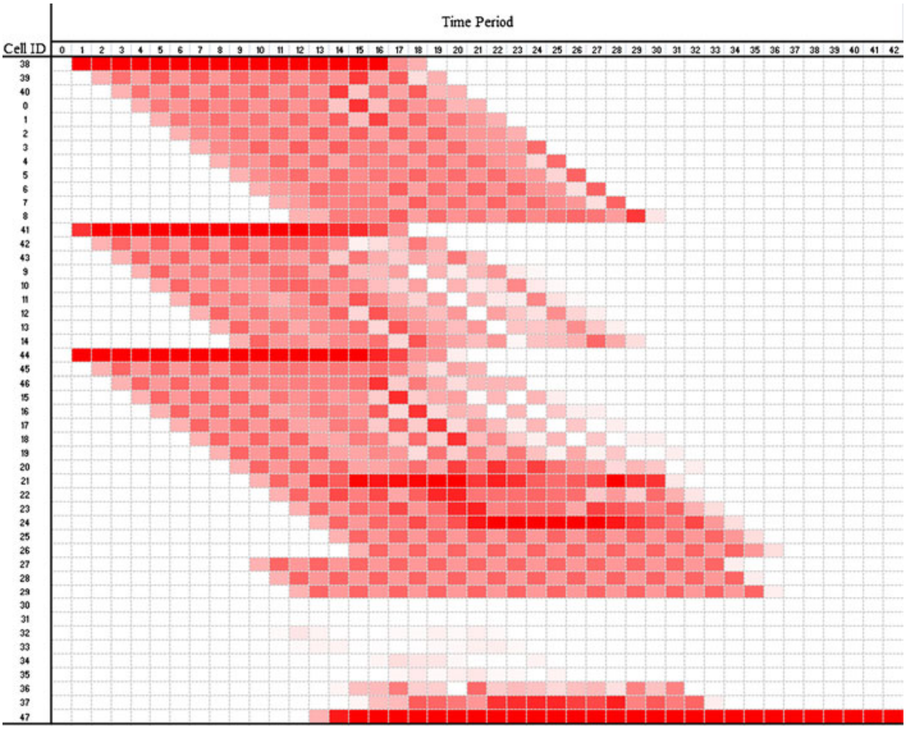

Fig. 3 Lightly loaded network, demand scenario 2

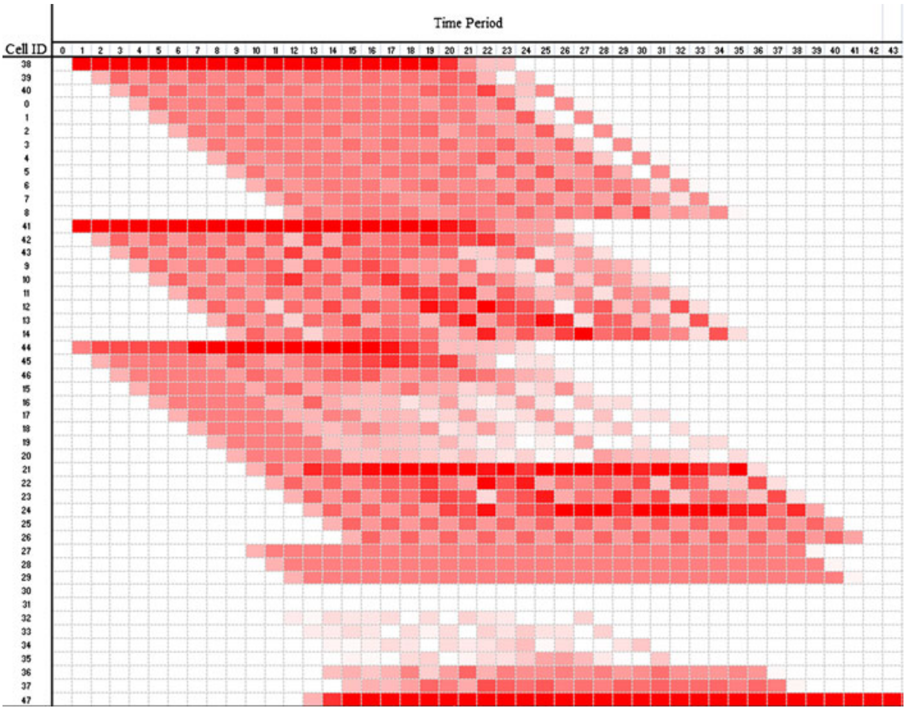

Fig. 4 Lightly loaded network, demand scenario 3

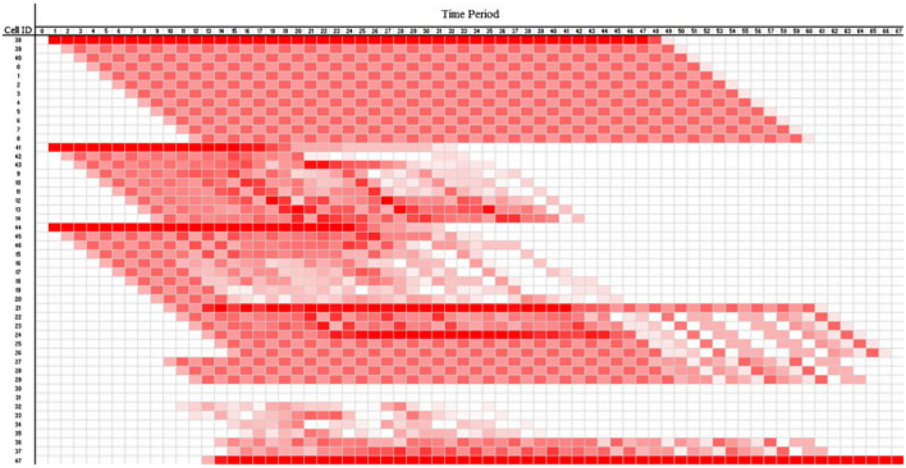

Fig. 5 Heavily loaded network, demand scenario 1

scenarios. The demand information for the lightly and heavily loaded networks are shown in Tables 4 and 5, respectively.

The LP formulation was prepared using C++ code, and solved using the CPLEX C++ API. For the lightly loaded network, 60 time periods were simulated so as to allow all flow to exit the network. For the heavily loaded network, 70 time periods were simulated for the same reason.

The resulting densities corresponding to the optimal path proportions for demand scenarios 1, 2, and 3 Figs. 2, 3 and 4 for the lightly loaded network, and Figs. 5, 6 and 7 for the heavily loaded network. The results are presented as a color-coded density matrix, meant to illustrate the time dependent densities patterns in the network; each row represents a cell, while each column represents a time period. Each cell's density

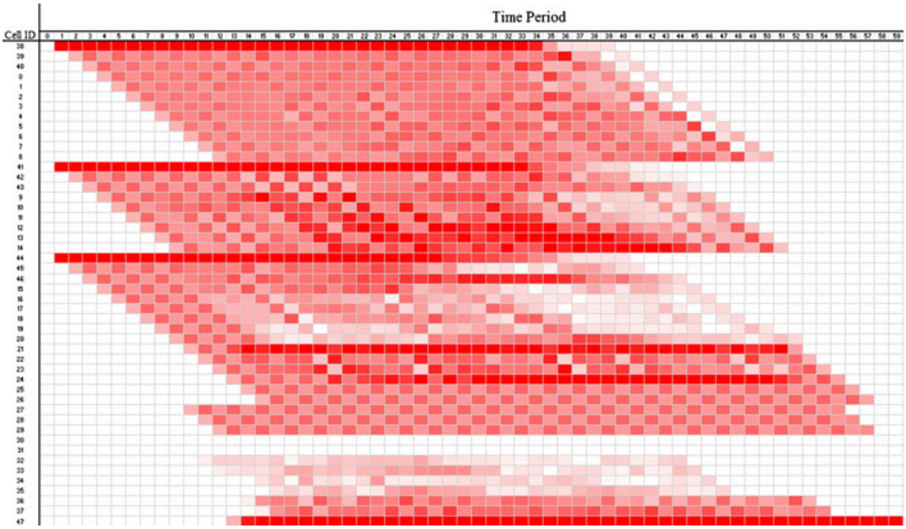

Fig. 6 Heavily loaded network, demand scenario 2

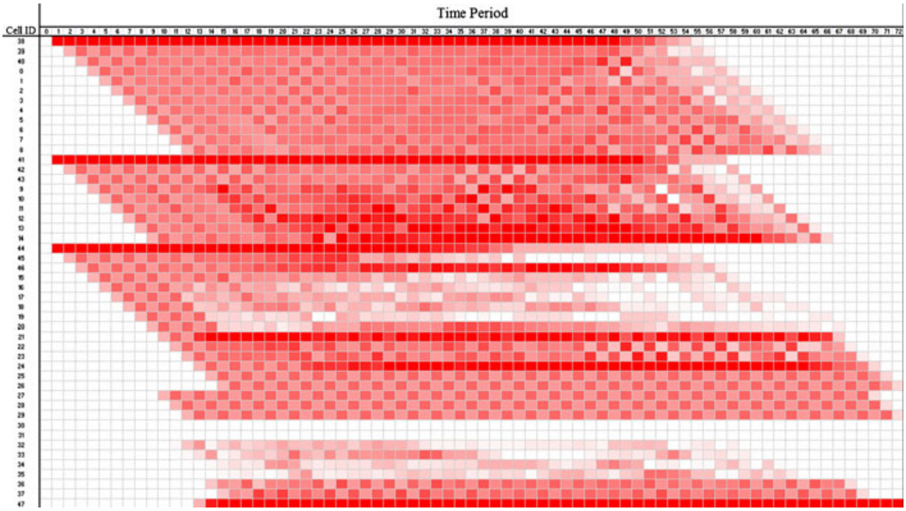

Fig. 7 Heavily loaded network, demand scenario 3

is represented by a color coded scale ranging from white (corresponding to a density of 0) to red (corresponding to a density of N_i). The colors were chosen so as to provide enough contrast to properly illustrate the flow patterns.

6 Discussion

While the linear programming formulation presented here is theoretically sound, the explicit enumeration of paths is concerning in regards to scalability. The number of variables increases in a manner directly proportionally to the number of paths, which in general grows combinatorially with respect to the network size. Therefore, the overall complexity of the problem as stated becomes very challenging when applied to realistically sized networks.

In spite of the drawbacks of the problem complexity, there are several research directions which could aid in the development of scalable solution methods for this problem. Two issues must be addressed in order to manage problem complexity: increased complexity with respect to the number of demand scenarios, and increased complexity due to path enumeration.

In order to manage the problem complexity with respect to the number of demand scenarios considered, decomposition methods, namely Dantzig-Wolfe decomposition techniques, lend themselves well for implementation on this problem. Because only the path proportion variables link the demand scenario-specific problems, implementations of Dantzig-Wolfe decomposition akin to that of Li et al. (2003) show significant promise.

Managing the enumeration of paths also presents several potential avenues of research. On one hand, column generation methods are well documented as options for generating good solutions to problems in which path enumeration may be otherwise required. Furthermore, other alternatives can be conceived based on the extraction of time-dependent paths from non-path based DTA approaches such as that presented in Ziliaskopoulos (2000). While the problem of extracting such paths is not trivial, the computational advantages of avoiding the solving of a path-based linear program may be significant.

7 Conclusions

In this paper, we introduced a novel strategic dynamic traffic assignment model that examines user responses to uncertain demand. Specifically, demand uncertainty is modeled in a scenario-based framework, and users react to the uncertainty by generating strategic strategies prior to the observation of the network demand (but with knowledge of the set of demand scenarios). The model is formulated as a linear program, based on a path-centric representation of flow. Substantial formulaic development was presented to enhance previously known DTA modeling techniques.

Numerical results were shown for a small network for demonstrative purposes. Cell and link variation across demand scenarios was analyzed, and computational complexity discussed. While the problem formulation and solution method are correct and sound, there are significant limitations regarding scalability due to the size of the linear program solved; both the number of demand scenarios considered and the path-based nature of the formulation severely affect the computational complexity of the problem.

Future research directions focused on developing scalable versions of the problem were also discussed. Decomposition methods are conceptually promising in reducing the overall size of the LP solved, while path generation and path-flow extraction techniques could curb the effect of the path-based formulation on complexity.

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